

On de Sitter-like and Minkowski-like space-times

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Abstract

Friedrich's proofs for the global existence results of de Sitter-like space-times and of semi-global existence of Minkowski-like space-times [Comm. Math. Phys. **107**, 587 (1986)] are re-examined and discussed, making use of the extended conformal field equations and a gauge based on conformal geodesics. In this gauge the location of the conformal boundary of the space-times is known *a priori* once the initial data has been prescribed. Thus it provides an analysis which is conceptually and calculationally simpler.

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1 Introduction

In [11, 6] the existence and stability of vacuum de Sitter-like space-times has been discussed. Moreover, reference [11] provides a semi-global existence and stability result for the development of hyperboloidal initial data which is close to Minkowski data. These results were subsequently generalised to the case where the gravitational field is coupled to Maxwell and Yang-Mills fields in [12]. These results make use of the Einstein conformal field equations—see e.g. [8, 9, 7, 10]—to reformulate Cauchy problems which are global or semi-global in time into problems which are local in time. Given one of these local Cauchy problems, then using powerful results of the theory of quasi-linear symmetric hyperbolic systems, e.g. [20, 21, 22], it is possible to prove the existence of solutions which are close to some explicitly known reference solutions—the de Sitter space-time and the Minkowski space-time. This particular strategy to prove global and semi-global existence and stability only works in 4 dimensions, although it should be mentioned that alternative proofs [1, 2] have been obtained using the so-called Fefferman-Graham conformal invariants which are valid in arbitrary even dimensions.

When discussing the conformal structure of space-times it can prove valuable to make use of gauges based on conformal invariants. One of these invariants, the conformal geodesics, has been introduced in [18] as a tool for the local analysis of the structure of conformally rescaled space-times. These conformal geodesics are associated to conformal structures in a similar way as geodesics are related to a metric. More importantly, conformal geodesics retain their character upon conformal rescalings. As in the case of the usual Gaussian coordinates, coordinates on a fiducial spacelike hypersurface are kept constant along a fixed congruence of timelike conformal geodesics. In [16] it has been shown that on the Schwarzschild-Kruskal space-time it is possible to construct a system of globally defined conformal Gaussian coordinates.

In [13] a general set of conformal field equations has been derived: *the extended conformal Einstein field equations*. These conformal equations are expressed using a so-called *Weyl or conformal connection*. A Weyl connection is a torsion free connection (not necessarily Levi-Civita)

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which preserves the conformal metric and for which parallel transport preserves conformally orthonormal frames and the causal nature of their vectors—that is, time-like vectors are transported into time-like vectors, etc. The extra freedom introduced by the use of Weyl connections allows one to consider gauges based on conformal geodesics in the discussion of various local and global issues in General Relativity. For example, in [13] this type of gauge and the evolution equations implied by them have been used to discuss the (local) existence of anti de Sitter-like space-times; in [14] conformal geodesics have been used to construct a representation of spatial infinity which makes possible, in principle, the detailed discussion of the structure of the gravitational field in this region of space-time.

One of the properties of conformal Gaussian coordinates is that in the case of vacuum space-times, they provide a canonical conformal factor, written entirely in terms of quantities defined on a fiduciary hypersurface and the conformal time parameter. In turn, this conformal factor provides *a priori* knowledge of the structure of the conformal boundary of the space-time. In this article we make use of this aspect to re-examine and discuss the global and semi-global existence results of [11]. The possibility of following this approach is somehow implicit in the literature, but to our knowledge has never been made explicit. The approach discussed here is in a sense more natural as it exploits the full freedom contained in the extended conformal field equations by making use of a gauge based on conformal invariants. In the original approach used in [11] the conformal factor is itself an unknown satisfying a propagation equation. Consequently the properties of the conformal boundary of the resulting space-time have to be discussed *a posteriori* in an abstract way. For example, in the case of the development of hyperboloidal data, the existence of a point corresponding to future time-like infinity where the generators of null infinity meet is inferred from a qualitative argument involving the Raychaudhuri equations. In contrast, in the approach followed here the existence and properties of future time-like infinity follow from explicit calculations and the requirement that the data is close to Minkowski data. Moreover, this information is available without reference to the existence problem of the propagation equations.

1.1 Structure and overview of the article

This article is structured as follows. Section 2 discusses congruences of conformal geodesics in vacuum space-times and the construction of conformal Gaussian coordinates on space-times whose time slices are diffeomorphic to the 3-sphere, or subsets thereof. The canonical conformal factors associated to the congruence of conformal geodesics are also studied. Section 3 is concerned with the conformal de Sitter and Minkowski space-times as subsets of the Einstein cylinder and with congruences of conformal geodesics on these manifolds. These space-times are analysed in detail as they will be used as reference solutions later on. Section 4 discusses the conformal boundary of the development of initial data sets which are sufficiently close to Cauchy data for the de Sitter space-time or to hyperboloidal data for the Minkowski space-time. Section 5 is concerned with the propagation equations implied by the extended conformal equations and the conformal Gaussian coordinates. The propagation equations are expressed in terms of a space spinor formalism. In section 6 the de Sitter and Minkowski space-times are recast as solutions of the propagation equations of section 5. Section 7 discusses the global and semi-global existence results implied by the propagation equations of section 5. These existence results are a consequence of a modified version of an existence and stability result for quasilinear hyperbolic systems by T. Kato. This modified Kato theorem was first discussed in [11] and is included here for completeness (theorem 1). The existence results considered here are: global existence for de Sitter-like space-times when data are given either on a standard Cauchy hypersurface (theorem 2) or on the conformal boundary (theorem 3); and semi-global existence for hyperboloidal Minkowski-like data (theorem 4). Section 8 provides some concluding remarks about possible generalisations of the results provided in the present work. An alternative discussion of the behaviour of conjugate points in the congruence of conformal geodesics is provided in the appendix.

2 Setup and gauge considerations

Let $(\tilde{\mathcal{M}}, \tilde{g}_{\mu\nu})$ be a space-time satisfying the Einstein field equations with cosmological constant

$$\tilde{R}_{\mu\nu} = \lambda \tilde{g}_{\mu\nu}.$$

We will only be concerned with the case $\lambda \leq 0$. Space-times such that $\lambda < 0$ and suitably close to the de Sitter solution will be called *de Sitter-like* whereas if $\lambda = 0$ and the space-time is suitably close to Minkowski space-time it will be called *Minkowski-like*. The metric $\tilde{g}_{\mu\nu}$ is assumed to have signature $(+, -, -, -)$.

2.1 Coordinatising manifolds diffeomorphic to \mathbb{S}^3

We shall work with space-times which are of the form $I \times \mathcal{S}$ where I is an interval on \mathbb{R} and \mathcal{S} is diffeomorphic to \mathbb{S}^3 or to a submanifold thereof. The manifold \mathbb{S}^3 will always be thought of as the following submanifold of \mathbb{R}^4 :

$$\mathbb{S}^3 = \left\{ x^{\mathcal{A}} \in \mathbb{R}^4 \mid (x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2 = 1 \right\}.$$

The restrictions of the functions $x^{\mathcal{A}}$, $\mathcal{A} = 1, 2, 3, 4$ on \mathbb{R}^4 to \mathbb{S}^3 will again be denoted by $x^{\mathcal{A}}$. The vector fields

$$c_1 \equiv x^1 \partial_4 - x^4 \partial_1 + x^2 \partial_3 - x^3 \partial_2, \quad (1a)$$

$$c_2 \equiv x^1 \partial_3 - x^3 \partial_1 + x^4 \partial_2 - x^2 \partial_4, \quad (1b)$$

$$c_3 \equiv x^1 \partial_2 - x^2 \partial_1 + x^3 \partial_4 - x^4 \partial_3. \quad (1c)$$

on \mathbb{R}^4 are tangent to \mathbb{S}^3 . In the sequel they will always be considered as vectors on \mathbb{S}^3 . Denote by $d\omega^2$ the line element obtained as pull-back of

$$\sum_{\mathcal{A}=1}^4 (dx^{\mathcal{A}})^2$$

to \mathbb{S}^3 . The vector fields c_1, c_2, c_3 constitute a globally defined frame on \mathbb{S}^3 which is orthonormal with respect to $d\omega^2$.

Let $\phi : \mathcal{S} \rightarrow \mathbb{S}^3$ denote the diffeomorphism connecting \mathcal{S} and \mathbb{S}^3 . The diffeomorphism ϕ will be employed to pull-back the functions $x^{\mathcal{A}}$, $\mathcal{A} = 1, 2, 3, 4$ on \mathbb{S}^3 to \mathcal{S} . Their pull-back to \mathcal{S} will again be denoted by the same symbol. This system of equations has rank 4 on \mathcal{S} , so that one can use a suitable choice of three of the functions $x^{\mathcal{A}}$, to obtain a coordinate system in a neighbourhood of a point in \mathcal{S} . Moreover, given $x^{\mathcal{A}}$ on \mathcal{S} , one can define the vectors c_1, c_2, c_3 as given by (1a)-(1c). In the following we will extend this frame by c_0 to a frame in \mathcal{M} . The index letters $\bar{s} = 0, 1, 2, 3$, respectively $\bar{r} = 1, 2, 3$, will be specifically reserved to denote components with respect to this frame.

2.2 Weyl connections and conformal geodesics

Let

$$g_{\mu\nu} = \Theta^2 \tilde{g}_{\mu\nu}$$

be a conformally related metric where Θ is some conformal factor. Let b_μ be a smooth 1-form. Denote by $\tilde{\nabla}$ and ∇ the Levi-Civita connections of \tilde{g} and g , respectively, and by $\hat{\nabla}$ the Weyl connection for \tilde{g} satisfying $\hat{\nabla} = \tilde{\nabla} + S(b)$, where

$$S(b)_\mu{}^\rho \equiv \delta^\rho{}_\mu b_\nu + \delta^\rho{}_\nu b_\mu - g_{\mu\nu} g^{\rho\lambda} b_\lambda.$$

Then $\nabla = \tilde{\nabla} + S(\Upsilon)$ with $\Upsilon = \Theta^{-1}\nabla\Theta$ and $\hat{\nabla} = \nabla + S(f)$ with $f = b - \Upsilon$. Further, define $d_\mu = \Theta b_\mu = \Theta f_\mu + \nabla_\mu \Theta$. The Schouten tensor associated to the Weyl connection $\hat{\nabla}$ is given by

$$\hat{L}_{\mu\nu} \equiv \frac{1}{2} \left(\hat{R}_{(\mu\nu)} - \frac{1}{2} \hat{R}_{[\mu\nu]} - \frac{1}{6} \hat{R}_{\lambda\rho} g^{\lambda\rho} g_{\mu\nu} \right).$$

We note that whenever the connection preserves a metric in the conformal class, i.e. it is a Levi-Civita connection, then $\hat{L}_{[\mu\nu]} = 0 = \hat{R}_{[\mu\nu]}$.

Let e_k , $k = 0, \dots, 3$, be a g -orthonormal frame field, satisfying $g(e_i, e_j) = \eta_{ij}$, with $\eta_{ij} \equiv \text{diag}(1, -1, -1, -1)$, $i, j = 0, \dots, 3$. Denote by ∇_k and $\hat{\nabla}_k$ the covariant derivative in the direction of e_k with respect to ∇ and $\hat{\nabla}$. Define the connection coefficients $\hat{\Gamma}_i^j{}_k$ of $\hat{\nabla}$ in this frame by $\hat{\nabla}_i e_k = \hat{\Gamma}_i^j{}_k e_j$. A *conformal geodesic* $x(\tau)$ is obtained, together with a 1-form $b(\tau)$ along the curve, as a solution to the system of equations

$$\begin{aligned} \dot{x}^\nu \tilde{\nabla}_\nu \dot{x}^\mu + S(b)_\lambda{}^\mu \dot{x}^\lambda \dot{x}^\mu &= 0, \\ \dot{x}^\nu \tilde{\nabla}_\nu b_\mu - \frac{1}{2} b_\nu S(b)_\lambda{}^\nu \dot{x}^\lambda &= \tilde{L}_{\lambda\mu} \dot{x}^\lambda, \end{aligned}$$

where \dot{x} denotes the tangent vector to the curve $x(\tau)$ and $\tilde{L}_{\mu\nu} = \tilde{\lambda} \tilde{g}_{\mu\nu}$ with $\lambda = 6\tilde{\lambda}$. In what follows we shall often write v^μ for \dot{x}^μ . Given initial data for these equations in the form $x_* \in \tilde{\mathcal{M}}$, $\dot{x}_* \in T_{x_*} \tilde{\mathcal{M}}$, $b_* \in T_{x_*}^* \tilde{\mathcal{M}}$, there exists a unique conformal geodesic $(x(\tau), b(\tau))$ near x_* satisfying for given $\tau_0 \in \mathbb{R}$

$$x(\tau_0) = x_*, \quad \dot{x}(\tau_0) = \dot{x}_*, \quad b(\tau_0) = b_*.$$

Conformal geodesics are conformally invariant in the sense that if $x(\tau)$ and $b(\tau)$ solve the conformal geodesics equations and we define a new Weyl connection $\tilde{\nabla} = \tilde{\nabla} + S(\tilde{b})$, then $(x(\tau), b(\tau) - \tilde{b}(\tau))$ solve the conformal geodesic equations with $\tilde{\nabla}$ replaced by $\tilde{\nabla}$ and $\tilde{L}_{\mu\nu}$ by $\tilde{L}_{\mu\nu}$. In particular if $\tilde{b} = b$ then $\tilde{\nabla}$ and $\tilde{\nabla}$ coincide and the conformal geodesic equations take the form

$$\begin{aligned} \dot{x}^\nu \hat{\nabla}_\nu \dot{x}^\mu &= 0, \\ \hat{L}_{\mu\nu} \dot{x}^\mu &= 0. \end{aligned}$$

2.3 Conformal Gaussian coordinates

Let $\tilde{\mathcal{S}}$ be a spacelike hypersurface in the vacuum space-time $(\tilde{\mathcal{M}}, \tilde{g})$. Let $\tilde{h}_{\alpha\beta}$ denote the intrinsic 3-metric of $\tilde{\mathcal{S}}$ induced by $\tilde{g}_{\mu\nu}$. On $\tilde{\mathcal{S}}$ choose:

- (i) a positive conformal factor Θ_* ;
- (ii) a frame field e_{*k} , $k = 0, \dots, 3$ such that $\tilde{g}(e_{*i}, e_{*k}) = \Theta_*^{-2} \eta_{ik}$,
- (iii) and a 1-form b_* .

Given the above initial information, there exists through each point $x_* \in \tilde{\mathcal{S}}$ a unique conformal geodesic $(x(\tau), b(\tau))$ with $\tau = \tau_*$ on $\tilde{\mathcal{S}}$ which satisfies the initial conditions $\dot{x}(\tau_*) = e_{0*}$, $b(\tau_*) = b_*$. These curves define a smooth-caustic free congruence in a neighbourhood \mathcal{U} of \mathcal{S} if all data are smooth. In addition, b defines a 1-form on \mathcal{U} from which one can construct a Weyl connection $\hat{\nabla} = \tilde{\nabla} + S(b)$. A smooth frame field e_k and a conformal factor Θ are then obtained on \mathcal{U} by solving the propagation equation

$$\dot{x}^\mu \hat{\nabla}_\mu e_k = 0.$$

It can be seen that $\tilde{g}(e_i, e_j) = \Theta^{-2} \eta_{ij}$ on \mathcal{U} with

$$\dot{x}^\mu \hat{\nabla}_\mu \Theta = \Theta \langle b, \dot{x} \rangle, \quad \Theta|_{\mathcal{S}} = \Theta_*, \quad (2)$$

where $\langle \cdot, \cdot \rangle$ denotes the contraction of a 1-form with a vector. Accordingly, the frame e_k is orthonormal for the metric $g_{\mu\nu} = \Theta^2 \tilde{g}_{\mu\nu}$. Coordinates x^A on $\tilde{\mathcal{S}}$ can be dragged along the congruence

of conformal geodesics, so that if one sets $x^0 = \tau$, one obtains a coordinatisation of $\mathcal{U} \subset \tilde{\mathcal{M}}$. In this gauge one has that

$$\dot{x} = v = e_0 = \partial_\tau, \quad \hat{\Gamma}_0^j{}_k = 0, \quad \hat{L}_{0k} = 0. \quad (3)$$

Such a choice of coordinates, frame field and conformal gauge will be referred to as a *conformal Gaussian system*. Let $g^{\mu\nu} = h^{\mu\nu} + v^\mu v^\nu$ where the pull-back of $h^{\mu\nu}$ is the contravariant negative definite intrinsic metric of the surfaces orthogonal to the congruence. The Levi-Civita connection of $h_{\mu\nu}$ will be denoted by D . Since $v = e_0$ is orthogonal to $\tilde{\mathcal{S}}$, $h_{\mu\nu}$ $\mu, \nu = 0, \dots, 3$, coincides with the 3-metric $h_{\alpha\beta} \equiv \Theta_*^2 \tilde{h}_{\alpha\beta}$ $\alpha, \beta = 1, 2, 3$. In general it will however not agree with the 3-metric of the surface of $\tau = \text{constant}$.

If \tilde{g} is a solution to the Einstein vacuum field equations with cosmological constant λ then for a conformal Gaussian system the conformal factor Θ and the 1-form d_k can be determined explicitly from equation (2) and the initial data. More precisely, one has that

$$\Theta_* \neq 0 \quad \Theta = \Theta_* \left(1 + \tau \langle b_*, \dot{x}_* \rangle + \frac{\tau^2}{2} \left(\tilde{\lambda} \Theta_*^{-2} + \frac{1}{2} g^\sharp(b_*, b_*) \right) \right), \quad (4a)$$

$$\Theta_* = 0 \quad \Theta = \langle d_*, v_* \rangle \tau + \frac{1}{2} \ddot{\Theta}_* \tau^2, \quad (4b)$$

and

$$d_0 = \dot{\Theta}, \quad d_a = \langle b_*, \Theta_* e_{a*} \rangle, \quad a = 1, 2, 3, \quad (5)$$

where we have set $b_* = \Upsilon_*$, $\tau_* = 0$ and used the identity

$$\ddot{\Theta} \Theta = \tilde{\lambda} + \frac{1}{2} g^\sharp(d, d), \quad (6)$$

In the above expressions, quantities with the subscript $*$ are regarded as constant along the conformal geodesics, while $g^\sharp(\cdot, \cdot)$ denotes the contravariant form of the metric $g_{\mu\nu}$ applied to a pair of 1-forms. As long as the congruence of conformal geodesics does not degenerate, then at the points where $\Theta = 0$, $\nabla_k \Theta \neq 0$ one finds that

$$g^\sharp(d, d) = \eta^{kl} \nabla_k \Theta \nabla_l \Theta = -2\tilde{\lambda}.$$

The nature of the conformal boundary (spacelike, timelike, null), defined by the conditions $\Theta = 0$, $\nabla_k \Theta \neq 0$, can be deduced according to whether $\tilde{\lambda}$ is negative, positive or zero, respectively.

2.4 Conjugate points

As mentioned in the previous paragraphs, it is necessary to check that the conformal geodesic congruence does not develop any conjugate points or caustics. This would lead to a break down of the conformal Gaussian coordinates. For our analysis we use the conformal Jacobi fields $\eta^\mu = \eta^k e_k^\mu$, as their component $h_{jk} \eta^k$ vanishes at a conjugate point [16]. They satisfy

$$\partial_\tau \eta_k = \chi_{jk} \eta^j, \quad (7a)$$

$$\partial_\tau^2 \eta_k = -(E_{jk} + \hat{L}_{jk}) \eta^j, \quad (7b)$$

$$\partial_\tau^3 \eta_k = -\partial_\tau (E_{jk}) \eta^j + \hat{Y}_{0jk} \eta^j, \quad (7c)$$

where χ_{jk} denotes the components, with respect to the frame e_k , of the second fundamental form of the surfaces of constant τ , and we have used the 4-dimensional Cotton-York tensor $\hat{Y}_{ijk} \equiv \hat{\nabla}_{[i} \hat{L}_{j]k}$, and the electric part of the Weyl tensor, $E_{ij} \equiv C_{0i0j}$.

3 The asymptotically simple model solutions

In this section we discuss the exact solutions of de Sitter ($\tilde{\lambda} = -1/2$) and Minkowski ($\tilde{\lambda} = 0$) in terms of conformal geodesics and initial data for the regular conformal field equations. It is well

known [19] that both solutions can be conformally embedded into the Einstein cylinder $\mathbb{R} \times \mathbb{S}^3$ with metric

$$g_E = d^2 t - d^2 \chi - \sin^2 \chi d\sigma^2,$$

where $\chi \in [0, \pi]$ and $d\sigma$ denotes the standard metric of \mathbb{S}^2 . The curves $x^\mu(t) = x_*^\mu + t\delta_0^\mu = x_*^\mu + tu^\mu$ are conformal geodesics written in terms of the coordinate t and $u = \partial_t$. Along these curves we have

$$\begin{aligned}\Theta_E(\tau) &= \left(1 + \frac{\tau^2}{4}\right), \\ v(\tau) &= \Theta_E(\tau)^{-1}u, \\ b_E(\tau) &= \frac{\tau}{2} dt = \frac{d\Theta_E}{\Theta_E},\end{aligned}$$

with

$$\tau \equiv 2 \tan \frac{t}{2},$$

and where Θ_E is the conformal factor generated along the geodesic by (2) with $\Theta_{E*} = 1$ at $t = 0$. Observe that $\tau \rightarrow \pm\infty$, $\Theta_E \rightarrow \infty$ and $v \rightarrow 0$ as $t \rightarrow \pm\pi$. The metric $g \equiv \Theta_E^2 g_E$ is given by

$$g = \Theta_E^2 g_E = d^2 \tau - \left(1 + \frac{\tau^2}{4}\right) (d^2 \chi + \sin^2 \chi d\sigma^2) \quad (8)$$

and satisfies $g(v, v) = 1$. Note that all curves are orthogonal to the surfaces of constant τ . In the rescaled space-time $b = b_E - \Theta_E^{-1} d\Theta_E = 0$, and thus, the curves are geodesics with respect to g .

3.1 The de Sitter space-time

The de Sitter space-time is embedded into the Einstein cylinder using the conformal factor

$$\Omega_D = \cos t = \left(\frac{1 - \frac{\tau^2}{4}}{1 + \frac{\tau^2}{4}}\right) = \frac{2 - \Theta_E}{\Theta_E}.$$

The conformal factor Ω_D vanishes at $\tau = \pm 2$, where the conformal boundary $\mathcal{J} = \mathcal{J}^- \cup \mathcal{J}^+$ is located. Using

$$\Upsilon_D = \frac{d\Omega_D}{\Omega_D} = -\frac{\tau}{1 - \frac{1}{4}\tau^2} d\tau,$$

setting $\Theta_{D*} = 1$ and using equation (2) we get that along the congruence of conformal geodesics

$$\begin{aligned}\Theta_D(\tau) &= 1 - \frac{1}{4}\tau^2 = \Omega_D \Theta_E, \\ b_D &= b_E + \Upsilon_D = \frac{-\frac{1}{2}\tau}{1 - \frac{1}{4}\tau^2} d\tau = \frac{d(\Omega_D \Theta_E)}{\Omega_D \Theta_E}.\end{aligned}$$

On the Cauchy surface $\tau = 0$ we have $b_D(0) = 0$. From equation (4a) we recover Θ_D , as above. We see that rescaling by Θ_D gives the metric g once more. The Cauchy surface $\tau = -2$ represents the past conformal boundary, \mathcal{J}^- . If we redefine $\tau \rightarrow \tau + 2$ we get on the conformal de Sitter space-time that

$$\begin{aligned}\Theta_D &= \tau - \frac{1}{4}\tau^2, \\ d_D &= \Theta_D b_D = -\frac{1}{2}(\tau - 2)d\tau, \\ v &= \frac{u}{1 + \frac{1}{4}(\tau - 2)^2},\end{aligned}$$

with the initial data at \mathcal{J}^- given by

$$d_{D*} = d\tau, \quad \langle d_D, v \rangle_* = 1, \quad \ddot{\Theta}_{D*} = -\frac{1}{2}.$$

We observe that in this case Θ_D is given by the formula (4b).

3.2 Minkowski space-time

The Minkowski space-time will be embedded using the conformal factor

$$\Omega_M = \cos\left(t + \frac{\pi}{2}\right) + \cos \chi.$$

For convenience we have shifted the standard embedding by $\pi/2$ to the past here, so that the usual Minkowskian hyperboloid, which is usually embedded at $t = \pi/2$, is now located at $t = 0$. We have

$$\Upsilon_M = \frac{d\Omega_M}{\Omega_M} = \frac{\cos t dt + \sin \chi d\chi}{\sin t - \cos \chi}.$$

It follows that the conformal geodesics satisfy

$$\begin{aligned} b_M &= b_E + \Upsilon_M = \tan\left(\frac{t}{2}\right) dt + \frac{\cos t dt + \sin \chi d\chi}{\sin t - \cos \chi}, \\ v &= \cos^2\left(\frac{t}{2}\right) u, \end{aligned}$$

Hence at $t = 0 = \tau$, we get the following information on the canonical Minkowski hyperboloid

$$\begin{aligned} \Omega_{M*} &= \cos \chi, \\ d_{M*} &= -(dt + \sin \chi), \\ \langle d_M, v \rangle_* &= -1, \\ h^\sharp(d, d)_* &= \sin^2 \chi. \end{aligned}$$

Substituting these into formula (4a) we get

$$\Theta_M = \cos \chi \left(1 - \frac{\tau}{\cos \chi} + \frac{\tau^2}{4}\right) = \Omega_D \Theta_E,$$

which vanishes at

$$\tau = 2 \frac{1 \pm \sin \chi}{\cos \chi}.$$

3.3 Conjugate points in the reference solutions

For the reference solutions discussed in this section the electric part of the Weyl tensor E_{jk} and the 4-dimensional Cotton-York tensor \hat{Y}_{ijk} vanish. Furthermore, the components of the second fundamental form $\chi_{jk}(0)$ vanishes. Using equations (7a)-(7c) one finds for the chosen congruence

$$\eta_0(\tau) = \eta_0(0), \quad \eta_k(\tau) = \Theta_E(\tau) \eta_k(0).$$

Now, $\Theta_E \neq 0$ for $\tau \in (-\infty, \infty)$. Thus, the Jacobi fields will never be tangent to the curve nor vanish. Hence the congruence is free of conjugate points.

4 The structure of the conformal boundary for nearby space-times

In this section we use the formulae (4a) and (4b) to study the conformal boundary of space-times which are constructed as the development of initial data which is close to either de Sitter Cauchy data or to hyperboloidal Minkowski initial data.

4.1 Space-times close to de Sitter

As discussed in [6], for de Sitter-like space-times one can formulate two slightly different Cauchy initial problems: one where data is prescribed on a standard Cauchy hypersurface, and a second one where the data is prescribed precisely on one portion of the conformal boundary, \mathcal{I}^- .

4.1.1 The case of de Sitter-like data away from \mathcal{I}^-

Here we assume that we are given a space-like hypersurface \mathcal{S} which does not intersect \mathcal{I}^- . So Θ is given by (4a) as $\Theta_* \neq 0$ on \mathcal{S} . In fact, without loss of generality we could set $\Theta_* = 1$. The conformal factor vanishes at

$$\tau_{\pm} = \frac{-2\Theta_* \langle d, v \rangle_* \pm 2\Theta_* \sqrt{|2\tilde{\lambda} + g^\sharp(d, d)_*|}}{2\tilde{\lambda} + g^\sharp(d, d)_*} \quad (9)$$

which gives the location of the conformal boundary, \mathcal{I}^\pm as smooth space-like hypersurfaces. If \mathcal{S} is topologically \mathbb{S}^3 —as it is being assumed here— then

$$\mathcal{I}^\pm = \{\tau_\pm\} \times \mathcal{S}$$

are topologically \mathbb{S}^3 . On \mathcal{I}^\pm we have $\nabla_k \Theta \nabla^k \Theta = -2\tilde{\lambda}$ and hence both hypersurfaces are space-like.

4.1.2 The case of de Sitter-like data on \mathcal{I}^-

We discuss now the data for a conformal geodesic congruence that starts on the smooth hypersurface \mathcal{S} which represents \mathcal{I}^- . Hence, the initial data satisfies the condition $\Theta_* = 0$ on \mathcal{S} and takes the form (4b). Equation (6) implies that $g^\sharp(d, d)_* = -2\tilde{\lambda}$ and thus d_* must be time-like as $\tilde{\lambda} < 0$. On the other hand, $\dot{\Theta}_*$ is free data on \mathcal{S} . Having set $b_* = \Upsilon_*$, that is, $d_* = (\nabla \Theta)_*$ and $v_* = n$, where n is the unit normal of \mathcal{S} with respect to g , it follows from $\Theta_* = 0$ on \mathcal{S} that

$$d_* = \langle d, v \rangle_* v_* \quad \text{with} \quad \dot{\Theta}_* = \langle d, v \rangle_* = \pm \sqrt{-2\tilde{\lambda}}.$$

We choose the positive root so that Θ is positive in the future of \mathcal{S} . Thus with respect to the Weyl propagated orthonormal frame e_k we obtain from 5

$$d_k(\tau) = \left(\sqrt{-2\tilde{\lambda}} + \ddot{\Theta}_* \tau, 0, 0, 0 \right).$$

The conformal factor vanishes at

$$\begin{aligned} \mathcal{I}^- &= \{0\} \times \mathcal{S}, \\ \mathcal{I}^+ &= \left\{ \tau = -2 \frac{\dot{\Theta}_*}{\ddot{\Theta}_*} \right\} \times \mathcal{S}. \end{aligned}$$

Hence, the location of \mathcal{I}^+ is determined by the free data $\ddot{\Theta}_*$. On \mathcal{I}^+ we have $d(\tau_+) = -d_*$ and $\nabla_k \Theta \nabla^k \Theta = -2\tilde{\lambda}$ and hence, again, it is a space-like hypersurface.

4.2 Hyperboloidal Minkowski data

We discuss now how to use formula (4a) to gain *a priori* information on the conformal boundary of the domain of dependence of hyperboloidal initial data which is close to Minkowski data. Given a 3-dimensional manifold \mathcal{S} with the topology of \mathbb{S}^3 , we consider $\overline{\mathcal{S}} \subset \mathcal{S}$, with $\partial \overline{\mathcal{S}}$ diffeomorphic to \mathbb{S}^2 . Furthermore, consider a function Ω on \mathcal{S} such that $\Omega > 0$ in the interior of $\overline{\mathcal{S}}$, and $\Omega = 0$ on $\mathcal{Z} \equiv \partial \overline{\mathcal{S}}$. The function Ω can be obtained naturally as part of a solution to the conformal Hamiltonian and momentum constraint — for a discussion and more details on this see [10, 11].

It is noted that

$$g^\sharp(b_*, b_*) = h^\sharp(b_*, b_*) + \langle b_*, v_* \rangle^2,$$

where $h^\sharp(b_*, b_*) \leq 0$, and $|h^\sharp(b_*, b_*)| = -h^\sharp(b_*, b_*)$ since $h_{\mu\nu}$ is taken to be negative definite. In order to make use of formula (4a), the following particular choices of initial data will be made

$$\Theta_* = \Omega, \quad \langle d_*, v_* \rangle = \dot{\Theta}_*, \quad d_{*a} = \Omega b_{*a} = D_a \Omega,$$

where $b_{*a} = e_a^\mu b_{*\mu}$, $D_a = e_a^\mu \nabla_\mu = e_a^\alpha D_\alpha$, $a = 1, 2, 3$. In particular, the sign of $\dot{\Theta}_*$ contains the information about which part of the locus of points such that $\Theta = 0$ should be considered as the conformal boundary —see the discussion below. On $\overline{\mathcal{S}} \setminus \mathcal{Z}$ the function $\dot{\Theta}_*$ is determined by (6) for the initial data $\dot{\Theta}_*$ is taken to extend smoothly to \mathcal{Z} . On \mathcal{Z} , the formula (6) implies that $g^\sharp(d, d)_* = g^\sharp(\nabla\Theta, \nabla\Theta)_* = 0$. For hyperboloidal data $(\nabla\Theta)_* \neq 0$ on \mathcal{Z} , and thus d_* must be null on \mathcal{Z} . We must thus have

$$\Xi^2 = \langle d_*, v_* \rangle^2 \quad \text{on} \quad \mathcal{Z},$$

where $\Xi \equiv \sqrt{|h^\sharp(d, d)_*|} = \sqrt{|D_k \Omega D^k \Omega|}$ on $\overline{\mathcal{S}}$. Consequently,

$$h^\sharp(b_*, b_*) = -\frac{4}{\omega^2}, \quad \text{with} \quad \omega \equiv \frac{2\Omega}{\Xi}.$$

The functions Ω , Ξ and ω will be extended off \mathcal{S} by requiring that they remain constant along a given curve of the congruence of conformal geodesics, and will be denoted again by Ω , Ξ , ω .

In order to further discuss the structure of the conformal boundary, we analyse the zeros of Θ . For curves passing through \mathcal{Z} the zeros are located at

$$\tau_\pm = \frac{-\dot{\Theta}_* \pm \dot{\Theta}_*}{\dot{\Theta}_*},$$

whereas on $\overline{\mathcal{S}} \setminus \mathcal{Z}$ we can write

$$\Theta = \Theta_* \left(1 + \alpha\tau + \left(\frac{1}{4}\alpha^2 - \frac{1}{\omega^2} \right) \tau^2 \right), \quad (10)$$

where $\alpha \equiv \langle b_*, v_* \rangle$. Note that α can be chosen independently of Ω —this fact will play a role when discussing the existence of solutions. The function α will be extended off $\overline{\mathcal{S}}$ in the same way as it was done for Ω , Ξ , ω . The roots of Θ are given by

$$\tau_\pm = \frac{-2\alpha\Omega^2 \pm 2\Omega\Xi}{\alpha^2\Omega^2 - \Xi^2}.$$

Accordingly, one defines

$$\mathcal{I}^\pm \equiv \left\{ (\tau, x^A) \in \mathbb{R} \times \mathcal{S} \mid \tau = \tau_\pm(x^A) \right\}. \quad (11)$$

This shows that the location of \mathcal{I}^\pm is predetermined by the initial data $\Omega, d_{*0}, D_k \Omega$ and well defined as long as the congruence does not degenerate. One sees that as $\Theta_* \rightarrow 0$ one has $\tau_\pm \rightarrow 0$, $-2\dot{\Theta}_*/\dot{\Theta}_*$. Thus, with the continuation of $\dot{\Theta}_*$ onto \mathcal{Z} described above one finds that \mathcal{I}^+ and \mathcal{I}^- are smooth hypersurfaces, whenever $d\Theta \neq 0$, and \mathcal{Z} is the intersection of \mathcal{I}^\pm with $\{0\} \times \mathcal{S}$ as expected for hyperboloidal initial data. In analogy to the model case of the hyperboloids in Minkowski space-time, the development of general hyperboloidal data has a conformal boundary which either corresponds to \mathcal{I}^+ or to \mathcal{I}^- , but not both. This information is contained in the sign of the free datum $\dot{\Theta}_*$. The conformal factor is positive on the physical space-time $(\tilde{\mathcal{M}}, \tilde{g}_{\mu\nu})$. So if $\dot{\Theta}_* > 0$ on \mathcal{Z} , then $\tilde{\mathcal{M}}$ lies to the future of \mathcal{I} . In this case one speaks of a hyperboloid which intersects past null infinity, and thus the conformal boundary is identified as \mathcal{I}^- . Whereas if $\dot{\Theta}_* < 0$ on \mathcal{Z} , then $\tilde{\mathcal{M}}$ lies to the past. Then the hyperboloid is regarded as intersecting future null infinity and then \mathcal{I}^+ gives the conformal boundary. Without loss of generality, in the sequel, we shall only consider hyperboloids intersecting future null infinity, so that $\dot{\Theta}_* < 0$ on \mathcal{Z} . Then \mathcal{I}^+ is given by τ_+ as identified above. We remark that the solution τ_- is of no interest to us as it lies outside the domain of dependence of $\overline{\mathcal{S}}$.

As $\overline{\mathcal{S}}$ is a compact set, there is a point in the interior of $\overline{\mathcal{S}}$ for which $D_i \Omega = 0$ and hence $\Xi = 0$. If the data is close enough to Minkowski data this critical point is unique. If one makes the choice $\alpha = 0$, then one finds that

$$\tau_\pm = \pm \frac{2\Omega}{\Xi} \longrightarrow \infty \quad \text{as} \quad \Xi \longrightarrow 0 \quad \text{while} \quad \Omega > 0.$$

Thus, in order to have a conformal representation of the domain of dependence of $\bar{\mathcal{S}}$ for which τ remains overall finite on \mathcal{I}^\pm one needs $\alpha \neq 0$.

In order to identify points which can be regarded as representing *timelike infinity*, one needs to investigate the critical points of Θ , that is, the points for which $d\Theta = 0$. One has that

$$\begin{aligned} d\Theta = & \left(1 + \alpha\tau + \left(\frac{1}{4}\alpha^2 - \frac{1}{\omega^2}\right)\tau^2\right) d\Omega \\ & + \Omega \left(\alpha + 2\tau \left(\frac{1}{4}\alpha^2 - \frac{1}{\omega^2}\right)\right) d\tau + \Omega\tau d\alpha + \Omega\tau^2 \left(\frac{1}{2}\alpha d\alpha + \frac{2}{\omega^3}d\omega\right). \end{aligned}$$

In particular, we are interested in analysing the conditions $d\Theta = 0$ on \mathcal{I}^\pm , with $\Omega \neq 0$. By construction one finds

$$\left(1 + \alpha\tau_\pm + \left(\frac{1}{4}\alpha^2 - \frac{1}{\omega^2}\right)\tau_\pm^2\right) = 0.$$

A necessary condition for the vanishing of $d\Theta$ on \mathcal{I}^\pm for $\Omega \neq 0$ is that

$$\alpha + 2\tau_\pm \left(\frac{1}{4}\alpha^2 - \frac{1}{\omega^2}\right) = 0.$$

A short calculation shows that the latter is equivalent to

$$\Xi^2 = D_k \Omega D^k \Omega = 0,$$

so that $D_k \Omega = 0$. Now, if $D_k \Omega = 0$ then $\tau_\pm = -2/\alpha$. Note that $\tau_\pm > 0$ if $\alpha < 0$ —that is, if $\Theta_* < 0$. Thus, in order to consider a conformal representation which includes the point i^+ , one needs to consider $\alpha \neq 0$. This condition will be assumed in the sequel. Let

$$\tau_{i^+} \equiv -2/\alpha.$$

From the discussion in section 3.2 it follows that in particular for the development of Minkowski data one has that $\tau_{i^+} = 2$.

Another computation shows that

$$\Omega\tau_{i^+} d\alpha + \Omega\tau_{i^+}^2 \left(\frac{1}{2}\alpha d\alpha + \frac{2}{\omega^3}d\omega\right) = 0$$

if $d\Omega = 0$. To show this, one uses the fact that if $D_k \Omega = 0$ and $\Omega \neq 0$ then it follows that $D_k(\Xi^2) = 0$ and furthermore, that $1/\omega = \Xi/2\Omega = 0$. In view of this, one defines $i^+ \in \mathbb{R} \times \mathcal{S}$ as the unique point for which $\tau = \tau_{i^+}$ and $d\Omega = 0$.

To conclude the discussion of the point i^+ , we look at the Hessian of Θ at i^+ , using the conformal Gaussian coordinates (τ, x^A) . Using $D_k \Omega = 0$ and $D_k(\Xi^2) = 0$ from above, one has

$$\nabla_{\mathcal{A}} \nabla_{\mathcal{B}} \left(\frac{1}{\omega^2}\right) = \frac{\nabla_{\mathcal{A}} \nabla_{\mathcal{B}} \Xi^2}{4\Omega^2} \quad \text{at } i^+.$$

Thus we find that

$$\begin{aligned} \nabla_{\mathcal{A}} \nabla_{\mathcal{B}} \Theta|_{i^+} &= \Omega \nabla_{\mathcal{A}} \nabla_{\mathcal{B}} \left(\alpha\tau + \left(\frac{1}{4}\alpha^2 - \frac{1}{\omega^2}\right)\tau^2\right) \\ &= \Omega\tau \nabla_{\mathcal{A}} \nabla_{\mathcal{B}} \alpha + \Omega\tau^2 \left(\frac{1}{2}\nabla_{\mathcal{A}} \alpha \nabla_{\mathcal{B}} \alpha + \frac{1}{2}\alpha \nabla_{\mathcal{A}} \nabla_{\mathcal{B}} \alpha - \frac{\nabla_{\mathcal{A}} \nabla_{\mathcal{B}} \Xi^2}{4\Omega^2}\right). \end{aligned}$$

Consequently

$$\nabla_{\mathcal{A}} \nabla_{\mathcal{B}} \Theta = \frac{2\Omega}{\alpha^2} \nabla_{\mathcal{A}} \alpha \nabla_{\mathcal{B}} \alpha - \frac{\nabla_{\mathcal{A}} \nabla_{\mathcal{B}} \Xi^2}{\Omega\alpha^2}, \quad \text{on } i^+. \quad (12)$$

Similar direct calculations render

$$\nabla_{\mathcal{A}}\nabla_0\Theta = \nabla_0\nabla_{\mathcal{A}}\Theta = -\Omega\nabla_{\mathcal{A}}\alpha, \quad \text{on } i^+, \quad (13a)$$

$$\nabla_0\nabla_0\Theta = \frac{1}{2}\Omega\alpha^2, \quad \text{on } i^+. \quad (13b)$$

Thus, if one chooses the function α such that $\alpha \neq 0$ (in order to have τ_{i^+} finite) and such that the 4×4 matrix with entries given by equations (12), (13a) and (13b) has no zero eigenvalues, it follows that the Hessian of Θ on i^+ is non-degenerate. Accordingly, the point i^+ can be rightfully regarded as the timelike infinity of the development of the hyperboloidal data prescribed on $\overline{\mathcal{S}}$.

5 Evolution equations

The general conformal Einstein field equations introduced in [13] —see also [15, 17]— are a generalisation of the original conformal equations which allows for the use of Weyl connections. The use of Weyl connections makes it possible to consider more general gauges when one is confronted with the need of deriving a system of propagation equations out of the conformal field equations. In particular, it makes it possible to make use of the conformal Gaussian coordinates discussed in the previous sections.

5.1 Evolution equations in a frame formalism

As shown in, for example [17], the general conformal field equations together with the gauge given by (3) imply the following system of propagation equations:

$$\begin{aligned} \partial_\tau e_{\bar{i}}^{\bar{s}} &= -\hat{\Gamma}_{\bar{i}}^{\bar{p}}{}_0 e_{\bar{p}}^{\bar{s}}, \\ \partial_\tau \hat{\Gamma}_{\bar{j}}^{\bar{k}}{}_{\bar{l}} &= -\hat{\Gamma}_{\bar{q}}^{\bar{k}} \hat{\Gamma}_{\bar{j}}^{\bar{q}}{}_0 + \delta_{\bar{q}}^{\bar{k}} \hat{L}_{\bar{j}\bar{l}} - \delta_{\bar{l}}^{\bar{k}} \hat{L}_{\bar{j}0} - g_{\bar{l}0} \hat{L}_{\bar{j}}^{\bar{k}} + \Theta d_{0\bar{j}}^{\bar{k}}{}_{\bar{l}}, \\ \partial_\tau \hat{L}_{\bar{j}\bar{l}} &= -\hat{\Gamma}_{\bar{j}}^{\bar{p}}{}_0 \hat{L}_{\bar{p}\bar{l}} + d_{\bar{k}} d_{0\bar{j}}^{\bar{k}}{}_{\bar{l}}, \\ \nabla_{\bar{k}} d_{\bar{i}\bar{j}}^{\bar{k}}{}_{\bar{l}} &= 0, \end{aligned}$$

where $d_{ijkl} = \Theta^{-1} C_{ijkl}$ denotes the components of the *rescaled Weyl tensor* with respect to the frame $e_{\bar{s}}$, while the conformal factor Θ and the 1-form d_k are given by (4a)-(5).

5.2 Evolution equations in a spinor formalism

In view of future applications, instead of the frame evolution equations discussed in the previous section, we shall use a spinorial version thereof. Using a spin dyad $\{o, \iota\}$ such that

$$\tau^{AA'} = \sqrt{2} v^{AA'} = o^A \bar{o}^{A'} + \iota^A \bar{\iota}^{A'},$$

the evolution equations take the form

$$\begin{aligned} \sqrt{2} \partial_\tau e_{CC'}^{\bar{s}} &= -\hat{\Gamma}_{CC'}^{EE'}{}_{DD'} \tau^{DD'} e_{EE'}^{\bar{s}}, \\ \sqrt{2} \partial_\tau \hat{\Gamma}_{DD'}^A{}_B &= -(\hat{\Gamma}_{DD'}^F{}_C \hat{\Gamma}_{FC'}^A{}_B + \bar{\Gamma}_{DD'}^F{}_C \hat{\Gamma}_{FC'}^A{}_B) \tau^{CC'} \\ &\quad + \hat{L}_{DD'BC'} \tau^{AC'} + \Theta \phi_{BCD}^A \tau^C{}_{D'}, \\ \sqrt{2} \partial_\tau \hat{L}_{AA'CC'} &= -(\hat{\Gamma}_{AA'}^F{}_B \hat{L}_{FB'CC'} + \bar{\Gamma}_{AA'}^{F'}{}_{B'} \hat{L}_{BF'CC'}) \tau^{BB'} \\ &\quad - d^{EE'} (\phi_{EABC} \epsilon_{E'C'} \tau_{A'}^B + \bar{\phi}_{E'A'B'C'} \epsilon_{EC} \tau_A^{B'}), \\ \sqrt{2} \partial_\tau \phi_{ABCD} &= \tau^{FF'} \nabla_{F'(D} \phi_{ABC)F} - \tau_{F'(D} \nabla^{FF'} \phi_{ABC)F}, \end{aligned}$$

where one has the correspondences (via the Infeld-van der Waerden symbols)

$$\begin{aligned}
e^{\bar{s}}_i &\mapsto e^{\bar{s}}_{CC'}, \\
\hat{\Gamma}^k_j &\mapsto \hat{\Gamma}^{AA'}_{BB'CC'} = \hat{\Gamma}^{AA'}_{BB'C} \epsilon_{C'}^{A'} + \bar{\hat{\Gamma}}^{AA'}_{BB'C'} \epsilon_C^A, \\
\hat{L}_{ij} &\mapsto \hat{L}_{AA'BB'}, \\
d_{ijkl} &\mapsto d_{AA'BB'CC'DD'} = \phi_{ABCD} \epsilon_{A'B'} \epsilon_{C'D'} + \bar{\phi}_{A'B'C'D'} \epsilon_{AB} \epsilon_{CD}, \\
d_j &\mapsto d_{AA'},
\end{aligned}$$

and the factors of $\sqrt{2}$ arise from the normalisation $\tau_{AA'} \tau^{AA'} = 2$.

5.3 Evolution equations in space-spinor formalism

Next, one introduces a *space-spinor formalism* by using the spinor $\tau_A^{A'}$ to eliminate all primed indices and then one splits the equations into symmetric and skew parts.

A space spinor $\Theta_{ABCD} = \Theta_{AB(CD)}$ is introduced and then decomposed such that

$$\Theta_{ABCD} \equiv \hat{L}_{CC'AA'} \tau_B^{C'} \tau_D^{A'} = \Theta_{(AB)(CD)} + \frac{1}{2} \epsilon_{AB} \Theta_G^G{}_{(CD)}.$$

For the spin coefficients $\hat{\Gamma}_{AA'BC}$, one observes that $\hat{\Gamma}_{AA'BC} = \Gamma_{AA'BC} + \epsilon_{AB} f_{CA'}$ and defines

$$\Gamma_{ABCD} \equiv \tau_B^{B'} \Gamma_{AB'CD},$$

which in turn, will be decomposed as

$$\Gamma_{ABCD} = \frac{1}{\sqrt{2}} (\xi_{ABCD} - \chi_{(AB)CD}) - \frac{1}{2} \epsilon_{AB} f_{CD}.$$

The spinors in the latter equation possess the following symmetries

$$\Gamma_{ABCD} = \Gamma_{AB(CD)}, \quad \chi_{ABCD} = \chi_{AB(CD)}, \quad \xi_{ABCD} = \xi_{(AB)(CD)}.$$

The term ξ_{ABCD} is related to the intrinsic connection of the leaves of the foliation defined by $\tau_{AA'}$; the term $\chi_{(AB)CD}$ to the second fundamental form of the leaves; and f_{AB} to the acceleration of the foliation. The spinors ξ_{ABCD} , $\chi_{(AB)CD}$ and f_{AB} are calculated from Γ_{ABCD} via the relations

$$\begin{aligned}
\xi_{ABCD} &= \frac{1}{\sqrt{2}} \left(\Gamma_{ABCD} + \tau_B^{B'} \tau_C^{C'} \tau_D^{D'} \bar{\Gamma}_{AB'C'D'} \right), \\
\chi_{ABCD} &= \frac{1}{\sqrt{2}} \left(\tau_B^{B'} \tau_C^{C'} \tau_D^{D'} \bar{\Gamma}_{AB'C'D'} - \Gamma_{ABCD} \right), \\
f_{AB} &= -\tau^{CC'} \Gamma_{CC'AB}.
\end{aligned}$$

The frame fields $e^{\bar{s}}_{AA'}$ are decomposed using

$$\begin{aligned}
e^0_{AA'} &= \frac{1}{\sqrt{2}} \tau_{AA'} - \tau^B_{A'} e^0_{AB}, \\
e^{\bar{r}}_{AA'} &= -\tau^B_{A'} e^{\bar{r}}_{AB},
\end{aligned}$$

with

$$e^{\bar{s}}_{AB} \equiv \tau_{(A}^{B'} e^{\bar{s}}_{B)B'}.$$

The fields $e^{\bar{s}}_{AB}$ are associated to spatial vectors, and hence, they satisfy the reality conditions

$$e^{\bar{s}}_{AB} = -\tau_A^{A'} \tau_B^{B'} \bar{e}^{\bar{s}}_{A'B'}. \quad (14)$$

Using the gauge given by (3) it can be shown that the extended conformal field equations given in [14] imply the following evolution equations for the unknowns $e_{AB}^{\bar{s}}$, ξ_{ABCD} , f_{AB} , $\chi_{(AB)CD}$, $\Theta_{(AB)CD}$, $\Theta_G^G{}_{CD}$:

$$\partial_\tau e_{AB}^0 = -\chi_{(AB)}^{EF} e_{EF}^0 - f_{AB}, \quad (15a)$$

$$\partial_\tau e_{AB}^{\bar{r}} = -\chi_{(AB)}^{EF} e_{EF}^{\bar{r}}, \quad (15b)$$

$$\begin{aligned} \partial_\tau \xi_{ABCD} = & -\chi_{(AB)}^{EF} \xi_{EFCD} + \frac{1}{\sqrt{2}} (\epsilon_{AC} \chi_{(BD)EF} + \epsilon_{BD} \chi_{(AC)EF}) f^{EF} \\ & - \sqrt{2} \chi_{(AB)(C}^E f_{D)E} - \frac{1}{2} (\epsilon_{AC} \Theta_F^F{}_{BD} + \epsilon_{BD} \Theta_F^F{}_{AC}) - i \Theta \mu_{ABCD}, \end{aligned} \quad (15c)$$

$$\partial_\tau f_{AB} = -\chi_{(AB)}^{EF} f_{EF} + \frac{1}{\sqrt{2}} \Theta_F^F{}_{AB}, \quad (15d)$$

$$\partial_\tau \chi_{(AB)CD} = -\chi_{(AB)}^{EF} \chi_{EFCD} - \Theta_{(CD)AB} + \Theta \eta_{ABCD}, \quad (15e)$$

$$\partial_\tau \Theta_{(AB)CD} = -\chi_{(CD)}^{EF} \Theta_{(AB)EF} - \partial_\tau \Theta \eta_{ABCD} + i \sqrt{2} d^E{}_{(A} \mu_{B)CDE}, \quad (15f)$$

$$\partial_\tau \Theta_G^G{}_{AB} = -\chi_{(AB)}^{EF} \Theta_G^G{}_{EF} + \sqrt{2} d^{EF} \eta_{ABEF}, \quad (15g)$$

where

$$\begin{aligned} \eta_{ABCD} &= \frac{1}{2} (\phi_{ABCD} + \tau_A^{A'} \tau_B^{B'} \tau_C^{C'} \tau_D^{D'} \bar{\phi}_{A'B'C'D'}), \\ \mu_{ABCD} &= -\frac{i}{2} (\phi_{ABCD} - \tau_A^{A'} \tau_B^{B'} \tau_C^{C'} \tau_D^{D'} \bar{\phi}_{A'B'C'D'}), \end{aligned}$$

denote, respectively, the electric and magnetic parts of ϕ_{ABCD} .

The evolution equations for the spinor ϕ_{ABCD} are derived from the Bianchi equations. Depending on the need, several alternative systems can be deduced. Here, we will consider the one which was called the *standard system* in references [13, 14]. Let

$$\phi_{ABCD} = \phi_0 \epsilon_{ABCD}^0 + \phi_1 \epsilon_{ABCD}^1 + \phi_2 \epsilon_{ABCD}^2 + \phi_3 \epsilon_{ABCD}^3 + \phi_4 \epsilon_{ABCD}^4,$$

where

$$\begin{aligned} \phi_i &\equiv \phi_{(ABCD)_i}, \quad i = 0, \dots, 4, \\ \epsilon_{ABCD}^0 &\equiv o_{(A} o_B o_C o_{D)}, \quad \epsilon_{ABCD}^1 \equiv \iota_{(A} o_B o_C o_{D)}, \quad \epsilon_{ABCD}^2 \equiv \iota_{(A} \iota_B o_C o_{D)}, \\ \epsilon_{ABCD}^3 &\equiv \iota_{(A} \iota_B \iota_C o_{D)}, \quad \epsilon_{ABCD}^4 \equiv \iota_{(A} \iota_B \iota_C \iota_{D)}. \end{aligned}$$

In the previous expressions the subindex $(ABCD)_i$ indicates that after the symmetrisation i indices are set to zero. One has the following *Bianchi propagation equations*:

$$\begin{aligned} (\sqrt{2} - 2e_{01}^0) \partial_\tau \phi_0 + 2e_{00}^0 \partial_\tau \phi_1 - 2e_{01}^{\bar{r}} \partial_{\bar{\tau}} \phi_0 + 2e_{00}^{\bar{r}} \partial_{\bar{\tau}} \phi_1 \\ = (2\Gamma_{0011} - 8\Gamma_{1010}) \phi_0 + (4\Gamma_{0001} + 8\Gamma_{1000}) \phi_1 - 6\Gamma_{0000} \phi_2, \end{aligned} \quad (16a)$$

$$\begin{aligned} \sqrt{2} \partial_\tau \phi_1 - e_{11}^0 \partial_\tau \phi_0 + e_{00}^0 \partial_\tau \phi_2 - e_{11}^{\bar{r}} \partial_{\bar{\tau}} \phi_0 + e_{00}^{\bar{r}} \partial_{\bar{\tau}} \phi_2 \\ = -(4\Gamma_{1110} + f_{11}) \phi_0 + (2\Gamma_{0011} + 4\Gamma_{1100} - 2f_{01}) \phi_1 + 3f_{00} \phi_2 - 2\Gamma_{0000} \phi_3, \end{aligned} \quad (16b)$$

$$\begin{aligned} \sqrt{2} \partial_\tau \phi_2 - e_{11}^0 \partial_\tau \phi_1 + e_{00}^0 \partial_\tau \phi_3 - e_{11}^{\bar{r}} \partial_{\bar{\tau}} \phi_1 + e_{00}^{\bar{r}} \partial_{\bar{\tau}} \phi_3 \\ = -\Gamma_{1111} \phi_0 - 2(\Gamma_{1101} + f_{11}) \phi_1 + 3(\Gamma_{0011} + \Gamma_{1100}) \phi_2 \\ - 2(\Gamma_{0001} - f_{00}) \phi_3 - \Gamma_{0000} \phi_4, \end{aligned} \quad (16c)$$

$$\begin{aligned} \sqrt{2} \partial_\tau \phi_3 - e_{11}^0 \partial_\tau \phi_2 + e_{00}^0 \partial_\tau \phi_4 - e_{11}^{\bar{r}} \partial_{\bar{\tau}} \phi_2 + e_{00}^{\bar{r}} \partial_{\bar{\tau}} \phi_4 \\ = -2\Gamma_{1111} \phi_1 - 3f_{11} \phi_2 + (2\Gamma_{1100} + 4\Gamma_{0011} + 2f_{01}) \phi_3 - (4\Gamma_{0001} - f_{00}) \phi_4, \end{aligned} \quad (16d)$$

$$\begin{aligned} (\sqrt{2} + 2e_{01}^0) \partial_\tau \phi_4 - 2e_{11}^0 \partial_\tau \phi_3 + 2e_{01}^{\bar{r}} \partial_{\bar{\tau}} \phi_4 - 2e_{11}^{\bar{r}} \partial_{\bar{\tau}} \phi_3 \\ = -6\Gamma_{1111} \phi_2 + (4\Gamma_{1110} + 8\Gamma_{0111}) \phi_3 + (2\Gamma_{1100} - 8\Gamma_{0101}) \phi_4. \end{aligned} \quad (16e)$$

5.4 Evolution equations for the Jacobi field

To the above propagation equations we will have to append an evolution equation for the Jacobi field. The conformal Jacobi field η^μ has a spinorial counterpart $\eta_{AA'}$ which can be split as

$$\eta_{AA'} = \frac{1}{2}\eta\tau_{AA'} - \tau_{A'}^B\eta_{AB},$$

with

$$\eta \equiv \eta_{AA'}\tau^{AA'}, \quad \eta_{AB} \equiv \tau_{(A}{}^{B'}\eta_{B)B'}.$$

Conjugate points in the congruence of conformal geodesics arise if $\eta_{AB} = 0$. The components η , η_{AB} satisfy the propagation equations

$$\sqrt{2}\partial_\tau\eta = f_{AB}\eta^{AB}, \quad (17a)$$

$$\sqrt{2}\partial_\tau\eta_{AB} = \chi_{CD(AB)}\eta^{CD}. \quad (17b)$$

5.5 Structural properties of the evolution equations

We discuss now some general structural properties of the equations (15a)-(15g), (16b)-(16e), (17a)-(17b) which will be used systematically in the sequel. Introduce the notation

$$v \equiv (e_{AB}^{\bar{s}}, \Gamma_{ABCD}, \Theta_{ABCD}), \quad \phi \equiv (\phi_0, \phi_1, \phi_2, \phi_3, \phi_4),$$

where it is understood that v contains only the independent components of the respective spinor—which are obtained by writing linear combinations of irreducible spinors, as discussed in the previous section. The unknown vector v has 45 independent complex components, while ϕ has 5 complex components. In terms of v and ϕ , the propagation equations (15a)-(15g) and (17a)-(17b) can be written as:

$$\partial_\tau v = Kv + Q(v, v) + L\phi, \quad (18)$$

where K and Q denote, respectively, a linear constant matrix-valued function and a bilinear vector-valued function both with constant entries and L is a linear matrix-valued function with coefficients depending on the coordinates. Similarly the system (16b)-(16e) can be written as

$$\sqrt{2}E\partial_\tau\phi + A^{AB}e_{AB}^{\bar{s}}\partial_{\bar{s}}\phi = B(\Gamma_{ABCD})\phi, \quad (19)$$

where E denotes the 5×5 identity matrix and $A^{AB}e_{AB}^{\bar{s}}$, $\bar{s} = 0, \dots, 3$, are 5×5 matrices depending on the coordinates, while $B(\Gamma_{ABCD})$ denotes a constant matrix-valued linear function of the connection coefficients Γ_{ABCD} . For later reference it is noted that

$$\sqrt{2}E + A^{AB}e_{AB}^0 = \begin{pmatrix} \sqrt{2} - 2e_{01}^0 & 2e_{00}^0 & 0 & 0 & 0 \\ -e_{11}^0 & \sqrt{2} & e_{00}^0 & 0 & 0 \\ 0 & -e_{11}^0 & \sqrt{2} & e_{00}^0 & 0 \\ 0 & 0 & -e_{11}^0 & \sqrt{2} & e_{00}^0 \\ 0 & 0 & 0 & -2e_{11}^0 & \sqrt{2} + 2e_{01}^0 \end{pmatrix},$$

and that

$$A^{AB}e_{AB}^{\bar{r}} = \begin{pmatrix} -2e_{01}^{\bar{r}} & 2e_{00}^{\bar{r}} & 0 & 0 & 0 \\ -e_{11}^{\bar{r}} & \sqrt{2} & e_{00}^{\bar{r}} & 0 & 0 \\ 0 & -e_{11}^{\bar{r}} & \sqrt{2} & e_{00}^{\bar{r}} & 0 \\ 0 & 0 & -e_{11}^{\bar{r}} & \sqrt{2} & e_{00}^{\bar{r}} \\ 0 & 0 & 0 & -2e_{11}^{\bar{r}} & 2e_{01}^{\bar{r}} \end{pmatrix},$$

with $\bar{r} = 1, 2, 3$. From the reality condition (14) one has that

$$\begin{aligned} e_{00}^0 &= -\bar{e}_{11}^0, & e_{00}^{\bar{r}} &= -\bar{e}_{11}^{\bar{r}}, \\ e_{01}^0 &= \bar{e}_{01}^0, & e_{01}^{\bar{r}} &= \bar{e}_{01}^{\bar{r}}, \end{aligned}$$

so that in particular e_{01}^0 and $e_{01}^{\bar{r}}$ are real. Strictly speaking, a discussion of the symmetric hyperbolicity of the system (18)-(19) should be carried out using real unknowns. In order to ease the presentation in the sequel, we write

$$e_{00}^{\bar{s}} = a^{\bar{s}} + ib^{\bar{s}}, \quad e_{01}^{\bar{s}} = c^{\bar{s}},$$

where $a^{\bar{s}}$, $b^{\bar{s}}$ and $c^{\bar{s}}$ denote the components of real vectors. Thus, making use of the splitting $\phi_j = \text{Re}(\phi_j) + i\text{Im}(\phi_j)$ and by multiplying the equations (16a)-(16e) by suitable numeric constants one finds that the 5×5 matrix $\sqrt{2}E + A^{AB}e_{AB}^0$ implies the 10×10 matrix:

$$\tilde{A}^0(a^0, b^0, c^0) = \begin{pmatrix} \frac{1}{\sqrt{2}} - c^0 & a^0 & 0 & 0 & 0 & 0 & -b^0 & 0 & 0 & 0 \\ a^0 & \sqrt{2} & a^0 & 0 & 0 & b^0 & 0 & -b^0 & 0 & 0 \\ 0 & a^0 & \sqrt{2} & a^0 & 0 & 0 & b^0 & 0 & -b^0 & 0 \\ 0 & 0 & a^0 & \sqrt{2} & a^0 & 0 & 0 & b^0 & 0 & -b^0 \\ 0 & 0 & 0 & a^0 & \frac{1}{\sqrt{2}} + c^0 & 0 & 0 & 0 & b^0 & 0 \\ 0 & b^0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} - c^0 & a^0 & 0 & 0 & 0 \\ -b^0 & 0 & b^0 & 0 & 0 & a^0 & \sqrt{2} & a^0 & 0 & 0 \\ 0 & -b^0 & 0 & b^0 & 0 & 0 & a^0 & \sqrt{2} & a^0 & 0 \\ 0 & 0 & -b^0 & 0 & b^0 & 0 & 0 & a^0 & \sqrt{2} & a^0 \\ 0 & 0 & 0 & -b^0 & 0 & 0 & 0 & 0 & a^0 & \frac{1}{\sqrt{2}} + c^0 \end{pmatrix}.$$

In particular note that if one sets $a^0 = b^0 = c^0 = 0$, then one gets

$$\tilde{A}^0(0, 0, 0) = \text{diag} \left(\frac{1}{\sqrt{2}}, \sqrt{2}, \sqrt{2}, \sqrt{2}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \sqrt{2}, \sqrt{2}, \sqrt{2}, \frac{1}{\sqrt{2}} \right). \quad (20)$$

Similarly, from the 5×5 matrix $A^{AB}e_{AB}^{\bar{r}}$ one deduces the (real) 10×10 symmetric matrix:

$$\tilde{A}^{\bar{r}}(a^{\bar{r}}, b^{\bar{r}}, c^{\bar{r}}) = \begin{pmatrix} -c^{\bar{r}} & a^{\bar{r}} & 0 & 0 & 0 & 0 & -b^{\bar{r}} & 0 & 0 & 0 \\ a^{\bar{r}} & \sqrt{2} & a^{\bar{r}} & 0 & 0 & b^{\bar{r}} & 0 & -b^{\bar{r}} & 0 & 0 \\ 0 & a^{\bar{r}} & \sqrt{2} & a^{\bar{r}} & 0 & b^{\bar{r}} & 0 & -b^{\bar{r}} & 0 & 0 \\ 0 & 0 & a^{\bar{r}} & \sqrt{2} & a^{\bar{r}} & 0 & 0 & b^{\bar{r}} & 0 & -b^{\bar{r}} \\ 0 & 0 & 0 & a^{\bar{r}} & c^{\bar{r}} & 0 & 0 & 0 & b^{\bar{r}} & 0 \\ 0 & b^{\bar{r}} & 0 & 0 & 0 & -c^{\bar{r}} & a^{\bar{r}} & 0 & 0 & 0 \\ -b^{\bar{r}} & 0 & b^{\bar{r}} & 0 & 0 & a^{\bar{r}} & \sqrt{2} & a^{\bar{r}} & 0 & 0 \\ 0 & -b^{\bar{r}} & 0 & b^{\bar{r}} & 0 & 0 & a^{\bar{r}} & \sqrt{2} & a^{\bar{r}} & 0 \\ 0 & 0 & -b^{\bar{r}} & 0 & b^{\bar{r}} & 0 & 0 & a^{\bar{r}} & \sqrt{2} & a^{\bar{r}} \\ 0 & 0 & 0 & -b^{\bar{r}} & 0 & 0 & 0 & 0 & a^{\bar{r}} & c^{\bar{r}} \end{pmatrix}.$$

For each $\bar{s} = 0, 1, 2, 3$, the matrices $\tilde{A}^{\bar{s}}(z)$ have entries which are polynomials of at most degree one in $z = (a^{\bar{s}}, b^{\bar{s}}, c^{\bar{s}})$. We can rewrite them in the following form

$$\tilde{A}^{\bar{s}}(z) = \tilde{A}^{\bar{s}}(0) + \check{\tilde{A}}^{\bar{s}}(z)$$

where $\tilde{A}^{\bar{s}}(0) \equiv \tilde{A}^{\bar{s}}(0, 0, 0)$ and $\check{\tilde{A}}^{\bar{s}}(x + y) = \check{\tilde{A}}^{\bar{s}}(x) + \check{\tilde{A}}^{\bar{s}}(y)$.

6 The conformal de Sitter and Minkowski space-times as solutions to the conformal field equations

In the conformal geodesic gauge given by (3) both the de Sitter and the Minkowski space-times are conformally rescaled to the unphysical space-time (\mathcal{M}, g) where $g = \Theta_E^2 g_E$ —see equation (8). The connection and curvature spinor components form the variables v and ϕ that satisfy the

equations (18) and (19) —respectively (15a)-(15g) and (16a)-(16e)— can be directly calculated from the components for the Einstein cylinder. More precisely, a straightforward calculation using the results of section 3 renders

$$e_{AB}^0 = 0, \quad (21a)$$

$$e_{AB}^{\bar{r}} = \frac{4}{\tau^2 + 4} \sigma_{AB}^{\bar{r}}, \quad (21b)$$

$$f_{AB} = 0, \quad (21c)$$

$$\xi_{ABCD} = \frac{-2i}{\tau^2 + 4} h_{ABCD}, \quad (21d)$$

$$\chi_{(AB)CD} = \frac{2\tau}{\tau^2 + 4} h_{ABCD}, \quad (21e)$$

$$\Theta_{ABCD} = \frac{-2}{\tau^2 + 4} h_{ABCD}, \quad (21f)$$

$$\phi_{ABCD} = 0, \quad (21g)$$

where $\sigma_{AB}^{\bar{r}} \equiv \sigma_{A'(A}^{\bar{r}} \tau_{B)}^{A'}$ are the spatial Infeld-van der Waerden symbols and

$$h_{ABCD} \equiv -\epsilon_{A(C} \epsilon_{D)B}.$$

The solutions to the conformal Jacobi equations (17a) and (17b) for the reference solution are given by

$$\eta = 0, \quad (22a)$$

$$\eta_{AB}^{\bar{r}} = \left(1 + \frac{\tau^2}{4}\right) \sigma_{AB}^{\bar{r}}. \quad (22b)$$

It is important to note that expressions (21a)-(21g) and (22a)-(22b) are valid for both the conformal de Sitter and the conformal Minkowski space-times. What distinguishes these two conformal space-times is the use of the appropriate conformal factor Θ_D , resp. Θ_M —cfr. section 3. It should also be observed that the expressions (21a)-(21g) and (22a)-(22b) are analytic functions of τ for $\tau \in \mathbb{R}$. Important in the sequel will be that $\eta_{AB}^{\bar{r}}$ are non-vanishing for $\tau \in (-\infty, \infty)$.

7 The existence and stability results

Having written the evolution equations in the conformal geodesic gauge, we proceed now to analyse the existence of solutions close to the explicit reference solutions of section 6. Following the original discussion in [11], the desired existence and stability results are obtained making use of slight modifications of very general theorems by Kato on the properties of symmetric hyperbolic systems [20, 21, 22].

7.1 Some further structural properties of the propagation system

Let $u = (\text{Re}(v), \text{Im}(v), \text{Re}(\phi), \text{Im}(\phi))$ with v and ϕ as in section 5.5. The unknown u takes values in \mathbb{R}^N for some $N \in \mathbb{N}$. The evolution equations (15a)-(15g) and (16a)-(16e) —or their matricial counterparts (18) and (19)— render a system of quasilinear partial differential equations for u which has the form

$$A^0(u) \cdot \partial_\tau u + \sum_{\bar{r}=1}^3 A^{\bar{r}}(u) \cdot c_{\bar{r}}(u) + B(\tau, x^A, u) \cdot u = 0, \quad (23)$$

with $c_{\bar{r}}(u)$ denoting the vector fields (1a)-(1c) acting on the unknown u . Furthermore,

$$A^0(u) = \begin{pmatrix} E & 0 \\ 0 & \tilde{A}^0(u) \end{pmatrix}, \quad A^{\bar{r}}(u) = \begin{pmatrix} 0 & 0 \\ 0 & \tilde{A}^{\bar{r}}(u) \end{pmatrix},$$

with $E, 0$ denoting, respectively, identity and zero matrices of the suitable dimensions and $\tilde{A}^0(u)$ and $\tilde{A}^{\bar{r}}(u)$ as given in section 5.5. In particular given any $z \in \mathbb{R}^N$, the matrix valued functions $A^{\bar{s}}(z)$, $\bar{s} = 0, 1, 2, 3$ have entries which are polynomial in z . These polynomials are at most of degree one and have constant coefficients. The matrices are symmetric ${}^t(A^{\bar{s}}(z)) = A^{\bar{s}}(z)$, $z \in \mathbb{R}^N$. The matrix valued function $B = B(\tau, x^{\mathcal{A}}, z)$ with $(\tau, x^{\mathcal{A}}, z) \in \mathbb{R} \times \mathcal{S} \times \mathbb{R}^N$, has entries which are polynomials in z with coefficients which are analytic functions on $\mathbb{R} \times \mathcal{S}$. Note that as \mathcal{S} is diffeomorphic to \mathbb{S}^3 , then B can be regarded as a matrix valued function with domain $\mathbb{R} \times \mathbb{S}^3 \times \mathbb{R}^N$ —this point of view will be adopted in what follows. These polynomials are at most of degree one. Following the decomposition in section 5.5, one can write

$$\begin{aligned} A^{\bar{s}}(z) &= A^{\bar{s}}(0) + \check{A}^{\bar{s}}(z), \\ B(\tau, x^{\mathcal{A}}, z) &= B(\tau, x^{\mathcal{A}}, 0) + \check{B}(\tau, x^{\mathcal{A}}, z), \end{aligned}$$

with

$$\begin{aligned} \check{A}^{\bar{s}}(y + z) &= \check{A}^{\bar{s}}(y) + \check{A}^{\bar{s}}(z), \\ \check{B}(\tau, x^{\mathcal{A}}, y + z) &= \check{B}(\tau, x^{\mathcal{A}}, y) + \check{B}(\tau, x^{\mathcal{A}}, z). \end{aligned}$$

Let u' denote the explicit reference solution given by (21a)-(21g). Set

$$u = u' + w. \quad (24)$$

This is in essence a definition for the new unknown w —which gives the perturbation from the reference solutions. Substitution of the Ansatz (24) into the system (23) yields

$$A^0(u' + w) \cdot \partial_\tau w + \sum_{\bar{r}=1}^3 A^{\bar{r}}(u' + w) \cdot c_{\bar{r}}(w) + B(\tau, x^{\mathcal{A}}, u' + w) \cdot w + \check{A}^0(w) \cdot \partial_\tau u' = 0,$$

where it has been used that

$$A^0(u') \cdot \partial_\tau u' + \sum_{\bar{r}=1}^3 A^{\bar{r}}(u') \cdot c_{\bar{r}}(u') + B(\tau, x^{\mathcal{A}}, u') \cdot u' = 0,$$

and that $c_{\bar{r}}(u') = 0$ —the reference solutions have no spatial dependence in the gauge being used. Thus, w satisfies

$$A^0(u' + w) \cdot \partial_\tau w + \sum_{\bar{r}=1}^3 A^{\bar{r}}(u' + w) \cdot c_{\bar{r}}(w) + \widehat{B}(\tau, x^{\mathcal{A}}, w) \cdot w = 0, \quad (25)$$

with $\widehat{B}(\tau, x^{\mathcal{A}}, z)$ again a matrix valued function with entries which are polynomials of at most degree one and coefficients which are analytic functions on $\mathbb{R} \times \mathbb{S}^3$, such that

$$\widehat{B}(\tau, x^{\mathcal{A}}, w) \cdot w = B(\tau, x^{\mathcal{A}}, u' + w) \cdot w + \check{A}^0(w) \cdot \partial_\tau u' + \check{B}(\tau, x^{\mathcal{A}}, w) \cdot u'.$$

We define new matrices $\mathcal{A}^0(w) = A^0(u' + w)$ and $\mathcal{A}^{\bar{r}}(w) = A^{\bar{r}}(u' + w)$ so that

$$\mathcal{A}^0(w) \cdot \partial_\tau w + \sum_{\bar{r}=1}^3 \mathcal{A}^{\bar{r}}(w) \cdot c_{\bar{r}}(w) + \widehat{B}(\tau, x^{\mathcal{A}}, w) \cdot w = 0, \quad (26)$$

For later use it is noted that the matrix $\mathcal{A}^0(u') = \mathcal{A}^0(0)$ is diagonal with constant entries. More precisely, it has the form

$$\mathcal{A}^0(0) = \begin{pmatrix} E & 0 \\ 0 & \tilde{A}^0(0) \end{pmatrix},$$

with E an identity matrix of the appropriate dimensions and $\tilde{A}^0(0)$ as given by equation (20). In particular, all the entries of $\mathcal{A}^0(u')$ are bigger or equal to $1/\sqrt{2}$.

7.2 The Kato existence and stability result

Let D and $d\mu$ denote, respectively, the Levi-Civita covariant derivative and the volume element associated with the standard metric on \mathbb{S}^3 and $D_{\bar{r}}$ denotes the covariant derivative in the direction of $c_{\bar{r}}$. On the space $C^\infty(\mathbb{S}^3, \mathbb{R}^N)$ of smooth \mathbb{R}^N -valued functions on \mathbb{S}^3 define for $m \in \mathbb{N}$ the Sobolev-like norm

$$\|w\|_m = \left(\sum_{k=0}^m \int_{\mathbb{S}^3} |D^k w|^2 d\mu \right)^{1/2}. \quad (27)$$

The notation

$$|D^0 w|^2 = |w|^2, \quad |D^k w|^2 = \sum_{\bar{r}_1, \dots, \bar{r}_k=1}^3 |D_{\bar{r}_1} \cdots D_{\bar{r}_k} w|^2,$$

is used, with $|\cdot|$ the standard Euclidean norm on \mathbb{R}^N . Given $m \in \mathbb{N}$ let $H^m(\mathbb{S}^3, \mathbb{R}^N)$ be the Hilbert space obtained as the completion of the space $C^\infty(\mathbb{S}^3, \mathbb{R}^N)$ in the norm (27). The unknown $w = w(\tau, x)$ will be regarded as a function of τ which takes values in $H^m(\mathbb{S}^3, \mathbb{R}^N)$. For $\delta \in \mathbb{R}$, $m \in \mathbb{N}$ with $0 < \delta < 1/\sqrt{2}$, $m \geq 2$, set

$$D_\delta^m = \left\{ w \in H^m(\mathbb{S}^3, \mathbb{R}^N) \mid (z, \mathcal{A}^0(w)z) > \delta(z, z), \forall z \in \mathbb{R}^N \right\},$$

where (\cdot, \cdot) is the standard scalar product on \mathbb{R}^N . Important for our purposes is that it contains a neighbourhood of the origin of $H^m(\mathbb{S}^3, \mathbb{R}^N)$ as the entries of $\mathcal{A}^0(0)$ are bounded from below by $1/\sqrt{2}$.

The original existence and stability results by Kato for symmetric hyperbolic systems of the form (26) which can be found in [22] —see also [20, 21]— have been worked out for the case when the frame fields $c_{\bar{r}}$ commute. In the case under discussion where the underlying leaves of the foliation of the space-time have the topology of \mathbb{S}^3 , the frame fields $c_{\bar{r}}$ do not commute. As discussed in [11], Kato's result can be modified to handle frame fields whose commutators are those of $O(3)$. For completeness we quote the result given in [11].

Theorem 1. *Suppose $m \geq 4$, D is a bounded open subset of $H^m(\mathbb{S}^3, \mathbb{R}^N)$ with $D \subset D_\delta^m$. If $w_0 \in D$ is given as initial condition for the system (26), then:*

- (i) *There exists a $T > 0$ and a unique solution $w(\tau)$ of equation (26) defined on $[0, T]$ with $w(0) = w_0$ and*

$$w \in C(0, T; D) \cap C^1(0, T; H^{m-1}(\mathbb{S}^3, \mathbb{R}^N)).$$

- (ii) *There is an $\varepsilon > 0$ such that one value for T can be chosen common to all initial conditions in the open ball $B_\varepsilon(w_0)$ with centre w_0 and radius ε , and such that $B_\varepsilon(w_0) \subset D$.*
- (iii) *If the solution $w(\tau)$ in (i) exists on $[0, T_0]$ for some $T_0 > 0$, then the solutions to all initial conditions in $B_\varepsilon(w_0)$ exist on $[0, T_0]$ if $\varepsilon > 0$ is sufficiently small.*
- (iv) *If ε and T are chosen as in (ii) and $w_0^n \in B_\varepsilon(w_0)$ with $\|w_0^n - w_0\|_m \rightarrow 0$ as $n \rightarrow \infty$, then for the solutions $w^n(\tau)$ with $w^n(0) = w_0^n$ it holds that $\|w^n(\tau) - w(\tau)\|_m \rightarrow 0$ uniformly in $\tau \in [0, T]$ as $n \rightarrow \infty$.*

Remark 1. The point (i) in the theorem establishes the local existence of solutions to the equation (26) for sufficiently small data (but not exclusively). By (ii) there is a non-vanishing existence time common to all solutions arising from data in a suitably small neighbourhood of the 0-data —the reference solution. In particular, by (iii), if a reference solution is known to have a certain existence time T_0 , then all solutions arising from data sufficiently close to the data of the reference solution have the same existence time. Finally point (iv) states that data close to a certain reference data gives rise to developments which are also close to the reference solution — i.e. stability.

Remark 2. A direct computation shows that on $[0, T]$ the solution to (26) is of class $H^m((0, T) \times \mathbb{S}^3) \subset C^{m-2}([0, T] \times \mathbb{S}^3)$. The convergence stated in (iv) is uniform on $[0, T] \times \mathbb{S}^3$.

7.3 Application to de Sitter-like space-times

7.3.1 Standard Cauchy data for de Sitter-like space-times

We start by considering the case where on a standard (spacelike) Cauchy hypersurface \mathcal{S} one is given de Sitter-like initial data. The initial data consists of the values of the spinorial fields $e_{AB}^{\bar{r}}$, f_{AB} , ξ_{ABCD} , $\chi_{(AB)CD}$, Θ_{ABCD} and ϕ_{ABCD} on the initial hypersurface \mathcal{S} . This information can be calculated from a solution to the conformal Hamiltonian and momentum constraints.

If the data are close to de Sitter data, then on \mathcal{S} (i.e. $\tau = 0$) one has that

$$e_{AB}^0 = 0 \quad (28a)$$

$$e_{AB}^{\bar{r}} = \sigma_{AB}^{\bar{r}}, \quad (28b)$$

$$f_{AB} = 0, \quad (28c)$$

$$\xi_{ABCD} = -\frac{i}{2}h_{ABCD} + \check{\xi}_{ABCD}, \quad (28d)$$

$$\chi_{(AB)CD} = \check{\chi}_{(AB)CD}, \quad (28e)$$

$$\Theta_{ABCD} = -\frac{1}{2}h_{ABCD} + \check{\Theta}_{ABCD}, \quad (28f)$$

$$\phi_{ABCD} = \check{\phi}_{ABCD}, \quad (28g)$$

where quantities with a $\check{}$ denote quantities which vanish for exactly de Sitter data.

Following the discussion in sections 3.1 and 4.1.1, initial data for the congruence of conformal geodesics will be chosen such that $\Theta_* = 1$, $b_* = 0$, and hence $\dot{\Theta}_* = \langle d, v \rangle_* = 0$. Upon this choice of data for the congruence, one has that the location of the conformal boundary of the development is given by (9) to be $\tau = \pm 2$ as in the case of the de Sitter space-time. Finally, the data (28a)-(28g) is supplemented by data for the Jacobi field $\eta_{AA'}$ of the form

$$\eta = 0, \quad \eta_{AB}^{\bar{r}} = \sigma_{AB}^{\bar{r}}. \quad (29)$$

From the previous discussion and theorem 1 one has the following existence and stability result.

Theorem 2. *Suppose $m \geq 4$. Let $u_0 = u'_0 + \check{u}_0$ be standard de Sitter-like Cauchy initial data. There exists $\varepsilon > 0$ such that if $\|\check{u}_0\|_m < \varepsilon$ then there is a unique solution $u' + \check{u}$ to the conformal propagation equations (15a)-(15g) and (16a)-(16e) with minimal existence interval $\tau \in [-2, 2]$ with $u \in C^{m-2}([-2, 2] \times \mathcal{S})$ and such that the associated congruence of conformal geodesics contains no conjugate points in $[-2, 2]$. The fields $u' + \check{u}$ imply a C^{m-2} solution to the vacuum Einstein field equations with positive cosmological constant for which the sets $\mathcal{I}^\pm = \{\pm 2\} \times \mathcal{S}$ represent future and past conformal infinity.*

Proof. Local existence to the system of the form (26) implied by the propagation equations (15a)-(15g), (16a)-(16e) and (17a)-(17b) follows from point (i) in theorem 1 and the observation that if $\varepsilon > 0$ is suitably small then $\check{u} \in D_\delta^m$. The reference solution given by (21a)-(21g) has existence interval $(-\infty, \infty) \supset [-2, 2]$, and the Jacobi fields associated to the congruence of conformal geodesics never vanish. Thus, from (ii) and (iii) in theorem 1 one has that for suitably small $\varepsilon > 0$ the developments of all data such that $\|\check{u}_0\|_m < \varepsilon$ have a minimum existence interval $[-2, 2]$. By reducing ε , if necessary, one can ensure that $\eta_{AB} \neq 0$ for $\tau \in [-2, 2]$ so that no conjugate points arise. \square

7.3.2 Asymptotic Cauchy data for de Sitter-like space-times

In the case of asymptotic Cauchy data for de Sitter-like space-times, whereby information is prescribed on an initial hypersurface \mathcal{S} which will be regarded as the past conformal infinity of the development, one has that the initial value of the fields $e_{AB}^{\bar{r}}$, f_{AB} , ξ_{ABCD} , $\chi_{(AB)CD}$, Θ_{ABCD} is calculated from the following fields on \mathcal{I}^- : a 3-metric $h_{\alpha\beta}$, χ^α_α and a symmetric trace-free

tensor field, $d_{\alpha\beta}$, satisfying $D^\alpha d_{\alpha\beta} = 0$, where D denotes the Levi-Civita connection of $h_{\alpha\beta}$. The form of the data for the propagation equations is formally identical to that of the data (28a)-(28g).

Following the discussion of sections 3.1 and 4.1.2 initial data for the congruence of conformal geodesics is chosen, without loss of generality, such that $\langle d, v \rangle_* = 1$ and $\dot{\Theta}_* = -1/2$. The data for the Jacobi field is taken to be identical to that in equation (29).

The corresponding existence, uniqueness and stability result for this case is the following.

Theorem 3. *Suppose $m \geq 4$. Let $u_0 = u'_0 + \check{u}_0$ be asymptotic de Sitter-like Cauchy initial data. There exists $\varepsilon > 0$ such that if $\|\check{u}_0\|_m < \varepsilon$ then there is a unique solution $u' + \check{u}$ to the conformal propagation equations (15a)-(15g) and (16a)-(16e) with minimal existence interval $\tau \in [0, 4]$ with $u \in C^{m-2}([0, 4] \times \mathcal{S})$ and such that the associated congruence of conformal geodesics contains no conjugate points in $[0, 4]$. The fields $u' + \check{u}$ imply a C^{m-2} solution to the vacuum Einstein field equations with positive cosmological constant for which the sets $\mathcal{I}^- = \{0\} \times \mathcal{S}$ and $\mathcal{I}^+ = \{4\} \times \mathcal{S}$ represent, respectively, past and future conformal infinity.*

Proof. The proof is identical to that of theorem 2.

7.4 Application to Minkowski-like space-times

In the case of hyperboloidal initial data one starts with a solution $(\Omega, \Sigma, h_{\alpha\beta}, \chi_{\alpha\beta})$ to the $\lambda = 0$ conformal Hamiltonian and momentum constraints:

$$\begin{aligned} 2\Omega D_\alpha D^\alpha \Omega - 3D_\alpha \Omega D^\alpha \Omega + \frac{1}{2}\Omega^2 r - 3\Sigma^2 - \frac{1}{2}\Omega^2 (\chi^2 - \chi_{\alpha\beta} \chi^{\alpha\beta}) + 2\Omega \Sigma \chi &= 0, \\ \Omega^3 D^\alpha (\Omega^{-2} \chi_{\alpha\beta}) - \Omega (D_\beta \chi - 2\Omega^{-1} D_\beta \Sigma) &= 0, \end{aligned}$$

where r denotes the Ricci scalar of the metric $h_{\alpha\beta}$ and $\chi = h^{\alpha\beta} \chi_{\alpha\beta}$. The solution to the conformal Hamiltonian and momentum constraints satisfies

$$\begin{aligned} \Omega = 0, \quad \Sigma < 0, \quad h^\sharp(D\Omega, D\Omega) = -\Sigma^2 &\quad \text{on } \mathcal{Z} \equiv \partial \overline{\mathcal{S}}, \\ \Omega > 0 &\quad \text{on } \overline{\mathcal{S}} \setminus \mathcal{Z}. \end{aligned}$$

The value of the fields ξ_{ABCD} , $\chi_{(AB)CD}$ are calculated from $(\Omega, \Sigma, h_{\alpha\beta}, \chi_{\alpha\beta})$. We set $\Theta_* = \Omega$, $\dot{\Theta}_* = \Sigma$ and e_{AB}^τ , f_{AB} as outlined before. The initial values of Θ_{ABCD} and ϕ_{ABCD} on the initial hyperboloid $\overline{\mathcal{S}}$ are then calculated from the above conformal constraint equations.

Theorem 1 gives existence for symmetric hyperbolic systems of the form (26) with data prescribed on an initial manifold, \mathcal{S} , which is topologically \mathbb{S}^3 . Consequently the data on $\overline{\mathcal{S}}$ has to be extended to data on the whole of \mathcal{S} . Noting that \mathcal{S} is diffeomorphic to \mathbb{S}^3 , we consider in what follows $\overline{\mathcal{S}}$ as a subset of \mathbb{S}^3 . As discussed in [11], there is a linear extension operator $E : H^m(\overline{\mathcal{S}}, \mathbb{R}^N) \longrightarrow H^m(\mathbb{S}^3, \mathbb{R}^N)$ such that if $v \in H^m(\overline{\mathcal{S}}, \mathbb{R}^N)$ then $(Ev)(x) = v(x)$ almost everywhere in $\overline{\mathcal{S}}$ and $\|Ev\|_m \leq K \|v\|_{m, \overline{\mathcal{S}}}$, with K a constant which is universal for fixed m . As in the cases of the de Sitter-like space-times, the data for the equation (26) takes the form $u_0 = u'_0 + \check{u}_0$. The vector u'_0 is defined as in equations (28a)-(28g) and thus it is defined on the whole of \mathcal{S} . On the other hand, the vector \check{u}_0 is only defined on $\overline{\mathcal{S}}$, and then needs to be extended. We define the extended data \hat{u}_0 by

$$\hat{u}_0 = u'_0 + E\check{u}_0.$$

It should be mentioned that the extension of the data is, in principle, non-unique and that in general \hat{u}_0 will not satisfy the conformal constraint equations on $\mathcal{S} \setminus \overline{\mathcal{S}}$. This will not have an effect on the development of the hyperboloidal data as $D^+(\overline{\mathcal{S}}) \cap I^+(\mathcal{S} \setminus \overline{\mathcal{S}}) = \emptyset$.

In the case of Minkowski-like data we shall make use of a conformal factor Θ of the form given by formula (10) with $\alpha < 0$. For a sufficiently small ball of data centred on Minkowski data this implies that the time τ_{i^+} is close to the one of Minkowski, namely $\tau_{i^+}^+ = 2$. The spatial location of i^+ is given by the condition $D_k \Omega = 0$ and will be close to that of i_M^+ .

The existence and stability result for Minkowski-like hyperboloidal data is the following.

Theorem 4. Suppose $m \geq 4$. Let $u_0 = u'_0 + \check{u}_0$ be Minkowski-like initial data. Given $T_0 > 2$ there exists $\varepsilon > 0$ such that

- (i) for $\|\check{u}_0\|_m < \varepsilon$ then there is a solution $u' + \check{u}$ to the conformal propagation equations (15a)-(15g) and (16a)-(16e) with minimal existence interval $\tau \in [0, T_0]$ and $u \in C^{m-2}([0, T_0] \times \mathcal{S})$;
- (ii) the associated congruence of conformal geodesics contains no conjugate points in $[0, T_0]$;
- (iii) for every $\check{u}_0 \in B_\varepsilon(0)$ there is a unique point in $\bar{\mathcal{S}} \setminus \mathcal{Z}$ such that $D_k \Omega = 0$;
- (iv) for all $\check{u}_0 \in B_\varepsilon(0)$ we have $\tau_{i^+} \in [0, T_0]$.

The solution $u' + \check{u}$ is unique on $\mathcal{D}^+(\bar{\mathcal{S}})$, the domain of dependence of $\bar{\mathcal{S}}$ and implies a C^{m-2} solution to the vacuum Einstein field equations with vanishing cosmological constant for which the set \mathcal{I}^+ as given by (11) represents future null infinity while the point i^+ given by the conditions $\tau = \tau_{i^+}$ and $D_k \Omega = 0$ represents timelike infinity.

Proof. As before, local existence to the system of the form (26) implied by the propagation equations (15a)-(15g) and (16a)-(16e) follows from point (i) in theorem 1 and the observation that if $\varepsilon > 0$ is suitably small then $u'_0 + E\check{u}_0 \in D_\delta^m$. The reference solution given by (21a)-(21g) has existence interval $(-\infty, \infty) \supset [0, T_0] \supset [0, -2/\alpha]$, and the Jacobi fields associated to the congruence of conformal geodesics never vanish. Note that T_0 is chosen independently of the initial data. Thus, from (ii) and (iii) in theorem 1 one has that for suitably small $\varepsilon > 0$ the developments of all extended data such that $\|E\check{u}_0\|_m < \varepsilon/K$ have a minimum existence interval $[0, T_0]$. By reducing ε , if necessary, one can ensure that $\eta_{AB} \neq 0$ for $\tau \in [0, T_0]$ so that no conjugate points arise on $\mathcal{S} \times [0, T_0]$. In particular, this also holds for $\mathcal{D}^+(\bar{\mathcal{S}}) \subset \mathcal{S} \times [0, T_0]$. The characterisation of the conformal boundary then follows from the discussion of section 4.2 \square

8 Conclusion and remarks

In this article we reinvestigated the problem of de Sitter-like and Minkowski-like space-times with the help of their conformal structure. The use of conformal Gaussian coordinates has several advantages. Their construction is conformally invariant and provides a natural conformal factor along the congruence, which for vacuum can be calculated explicitly. Furthermore, the associated Weyl connection yields a gauge in which the extended conformal field equations are simplified and a symmetric hyperbolic system is obtained. It was shown that for these vacuum space-times the location of the conformal boundary $\mathcal{I}^+ \cup i^+$ can be read off directly from the initial data. It should be mentioned that this formulation of the initial value problem for the conformal Einstein field equations is amenable to numerical implementations, and indeed, a frame version thereof has been used in the numerical investigations of cosmological space-times described in [3, 4].

For our calculations certain choices of initial data were motivated by the behaviour of the exact solution and their conformal embedding into the Einstein cylinder. Hence on the initial surface we set $b_k = \Upsilon_k$. It should be noted that not all congruences of conformal geodesics are suitable as they can develop conjugate points before or at the conformal boundary. A simple example is provided by the standard time-like geodesics in Minkowski space, which are conformal geodesics with $b_k = 0$. In the conformally compactified picture they converge at i^+ , where the congruence has a conformal conjugate point. This is the only point of the conformal boundary that is reached. Moreover, it takes infinite time τ to get there and the induced conformal factor is constant along each curve.

Other choices of initial data are related to the parametrisation of the congruence, e.g. for de Sitter-like space-times $\check{\Theta}_*$ is free datum on \mathcal{I}^- . These choices affect the location of \mathcal{I}^+ in our conformal Gaussian coordinates and as long as we avoid $\tau \rightarrow \infty$ before reaching \mathcal{I}^+ we can specify them suitably. However, these choices do not reduce the class of de Sitter-like space-times for which theorem 3 holds.

Finally, it is mentioned that the methods discussed here can be adapted to discuss the stability of other suitable reference solutions like the *purely radiative space-times* of [11]. The detailed discussion of this generalisation will be presented elsewhere.

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A An alternative discussion of the conjugate points

In the main part of the article the variable $u = (v, \phi)$ contained the Jacobi fields and thus guaranteed that one can avoid conjugate points. In this appendix we outline an alternative approach. Kato's theorem —cfr. theorem 1— can be used for an unknown u which does not contain η_{AB} . It states that one can find initial data close to the original data such that the components of the curvature tensors stay within certain bounds. This approach will be taken here. The Jacobi fields are here treated outside the symmetric hyperbolic system following work in [23].

Since the reference solution is conformally flat its Weyl tensor and Cotton-York tensor vanish. However the second fundamental form and the Schouten tensor do not vanish and hence it is more difficult to obtain suitable bounds for the estimations that are to follow. Hence we use the second order Jacobi equation and replace the Schouten tensor with the Cotton-York tensor in it. The conformal Jacobi equation (7b) is linear. Thus it is sufficient to consider initial data with

$$z(0) = 1, \quad \eta^\mu(0) = e_k^\mu, \quad \dot{z}(0) = \chi_{kk}.$$

Furthermore, let $X \equiv \dot{z}(0)$. We split the Jacobi fields into

$$\eta^\mu = \eta_0 v^\mu + z m^\mu$$

where $z^2 \equiv -h(\eta, \eta)$ and $h(m, m) = -1$. Denoting $d/d\tau$ by D , we can derive

$$\ddot{z} = -\frac{h(D^2\eta, \eta)}{z} + \frac{h(D\eta, D\eta)h(\eta, \eta) - (h(D\eta, \eta))^2}{z^3} \geq -\frac{h(D^2\eta, \eta)}{z} = (E_{ik} + \hat{L}_{ik})m^i m^k z,$$

where we have used the Cauchy-Schwartz inequality and the conformal Jacobi equation (7b). We observe the identity $D(\hat{L}(\eta, e_k)) = \hat{Y}(v, \eta, e_k)$ and its integral form

$$\int_0^\tau \hat{Y}_{0jk} \eta^j d\sigma = \hat{L}(\eta, e_k)(\tau) - \hat{L}(\eta, e_k)(0).$$

This allows us to replace the Schouten tensor in the above inequality and obtain

$$\ddot{z} \geq E(m, m)(\tau)z(\tau) + \int_0^\tau (\hat{Y}_{0jk} z m^j)(\sigma) m^k(\tau) d\sigma + (z \hat{L}_{ik} m^i)(0) m^k(\tau).$$

Observing that m is a space-like unit vector and employing a Cauchy-Schwartz type argument in combination with the existence and stability theorem (1), we can see that there exist bounds such that

$$-K \leq E(m, m)(\tau) \leq K \quad -L \leq \hat{Y}_{0jk} m^j(\sigma) m^k(\tau) \leq L.$$

For the Schouten tensor on the initial surface we use

$$-Q \leq \hat{L}_{ik} z m^i(0) m^k(\tau) \leq Q',$$

with $Q, Q' \geq 0$. This simplifies the inequality to give

$$\ddot{z} \geq -Kz(\tau) - L \int_0^\tau z(\sigma) d\sigma - Q.$$

One now follows the argument of [23], by introducing a family of functions $y_\varepsilon(\tau)$ satisfying

$$\ddot{y}_\varepsilon = -(K + \varepsilon^2)y_\varepsilon(\tau) - L \int_0^\tau y_\varepsilon(\sigma) d\sigma - Q \quad (30)$$

with initial data $y_\varepsilon(0) = 1$, $\dot{y}_\varepsilon(0) = X$. The idea is to show that $z(\tau)$ cannot vanish before $y_\varepsilon(\tau)$, in particular when $\varepsilon = 0$. By estimating a lower bound for the time τ_y when $y_{\varepsilon=0}$ vanishes, we get a lower bound for the time before the solution can develop conjugate points.

We briefly recall the steps in the argument, whose details can be found in [23, 5]. We drop the subscript ε . One defines $R \equiv z/y$ and $W \equiv \dot{z}y - \dot{y}z = y^2\dot{R}$. From this one derives

$$\dot{W}(\tau) \geq L \int_0^\tau y(\tau)y(\sigma)[R(\tau) - R(\sigma)]d\sigma + Qy(\tau)[R(\tau) - 1].$$

There must be an interval $J = [0, T_\varepsilon]$ such that $W \geq 0$ and hence that $R \geq 1$ and $\dot{W} > 0$ hold on J . This implies that as long as y does not vanish, we can extend J further while $R \geq 1$ holds. Hence $z \geq y \geq 0$ on J . By continuity in ε , $\lim_{\varepsilon \rightarrow 0} T_\varepsilon = T_0 > 0$. Thus we only need to focus on $y = y_{\varepsilon=0}$, for which integrate (30) twice to get

$$y(\tau) = 1 - X\tau - \frac{1}{2}Q\tau^2 - K \int_0^\tau (\tau - \sigma)y(\sigma)d\sigma - L \int_0^\tau \frac{1}{2}(\tau - \sigma)^2 y(\sigma)d\sigma.$$

Observing that $y(\tau) \leq y(0) + X\tau$ and using it to obtain upper bounds for the integrals we get

$$y(\tau) \geq Y(\tau) \equiv 1 - X\tau - \frac{1}{2}(Q + K)\tau^2 - \frac{1}{6}(XK + L)\tau^3 - \frac{LX}{24}\tau^4$$

For the reference solution all constants vanish so that $z \geq y \geq 1$ for all τ , which agrees with $z = 1 + \tau^2/4$. If one of the constants X, Q, K, L is non-zero, we get exactly one positive root, as all points of inflection have to lie in the second and third quadrant.

Without loss of generality we can set all of them to R by appealing to Kato's theorem. Suppose we now fix $\tau = T$ suitably beyond the values at future infinity of the space-time we would like to perturb. Then $F(T) = 1 - A(T)R - B(T)R^2$, for some constants $A(T), B(T)$. It is clear that we can always choose the perturbation of the initial data suitably small in Kato's theorem, so that the bound R guarantees that $f(T) > 0$ and hence that the perturbed solution will not develop conjugate points before the chosen time $\tau = T$.

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